Set Controllability, Observability and Output Tracking of Boolean Networks

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Outline

1. Introduction
2. Set Controllability of Boolean Networks
3. Observability of Boolean Networks
4. Output Tracking of Boolean Networks
5. Concluding Remarks
Introduction

Boolean Networks

With the rapid development of systems biology and medical science, Boolean networks (BNs) become an active research topic in biology, physics and engineering.

The control of BNs is important for the disease treatment and pharmaceutical preparation.*

A major goal of systems biology is to develop suitable mathematical tools for the analysis and control of complex biological systems.*

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Logical Networks

- **Classification** of logical networks: BNs, multi-valued logical networks, mix-valued logical networks.
- **Applications** of logical networks: circuit design, finite automata, game theory, graph theory, fuzzy control, feedback register, and so on.
- **Existing methods**: computer simulation, polynomial theory over finite field, semi-tensor product of matrices.
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Logical Networks
Consider the following Boolean model of signal transduction networks:

\[
\begin{align*}
  x_1(t+1) &= x_8(t), \\
  x_2(t+1) &= x_1(t), \\
  x_3(t+1) &= x_2(t), \\
  x_4(t+1) &= x_8(t), \\
  x_5(t+1) &= x_4(t), \\
  x_6(t+1) &= x_3(t) \lor x_5(t), \\
  x_7(t+1) &= x_8(t), \\
  x_8(t+1) &= x_6(t) \land \neg x_7(t),
\end{align*}
\]

where \(x_1\) stands for the nitric oxide synthase (NOS), \(x_2\) represents the nitric oxide (NO), \(x_3\) is the guanyl cyclase (GC), \(x_4\) is the phospholipase C (PLC), \(x_5\) represents the inositol-1,4,5-trisphosphate (InsP3), \(x_6\) is the \(\text{Ca}^{2+}\) influx to the cytosol from intracellular stores (CIS), \(x_7\) stands for the \(\text{Ca}^{2+}\) ATPases and \(\text{Ca}^{2+}/\text{H}^{+}\) antiporters responsible for \(\text{Ca}^{2+}\) efflux from the cytosol (\(\text{Ca}^{2+}\) ATPase), and \(x_8\) is the cytosolic \(\text{Ca}^{2+}\) increase (\(\text{Ca}^{2+}\_c\)).

Logical Networks

The "minimal" Boolean model for the lactose operon in *Escherichia coli* is given as follows:

\[
\begin{align*}
    x_1(t+1) &= \neg u_1(t) \land (x_3(t) \lor u_2(t)), \\
    x_2(t+1) &= x_1(t), \\
    x_3(t+1) &= \neg u_1(t) \land [(x_2(t) \land u_2(t)) \lor (x_3(t) \land \neg x_2(t))] 
\end{align*}
\]

where \( x_1 \in \mathcal{D} \) denotes the mRNA, \( x_2 \in \mathcal{D} \) the lacZ polypeptide, \( x_3 \in \mathcal{D} \) the intracellular lactose, \( u_1 \in \mathcal{D} \) the external glucose, and \( u_2 \in \mathcal{D} \) the external lactose.

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Logical Networks

Consider a networked evolutionary game (NEG) consisting of four players, in which the set of players is denoted by $N = \{P_1, P_2, P_3, P_4\}$ and the network graph of the game is string. The neighborhood of each $P_i$ is denoted by $U(i)$. The basic game of this NEG is the Rock-Scissors-Paper game, whose payoff matrix is given in Table 1, where “Rock”, “Scissors” and “Paper” are denoted by “1”, “2” and “3”, respectively. Hence, all the players have the same set of strategies: $S = \{1, 2, 3\}$.

<table>
<thead>
<tr>
<th>$P_1$ \ $P_2$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0)</td>
<td>(1, -1)</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(-1, 1)</td>
<td>(0, 0)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>3</td>
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<td>(-1, 1)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>
Introduction

Logical Networks

Suppose that the game can repeat infinitely. At each time, $P_i$ only plays the Rock-Scissors-Paper game with its neighbors in $U(i)$, and its aggregate payoff $c_i : S^{\mid U(i) \mid} \rightarrow \mathbb{R}$ is the sum of payoffs gained by playing with all its neighbors in $U(i)$, that is,

$$c_i(P_i, P_j | j \in U(i)) = \sum_{j \in U(i)} c_{ij}(P_i, P_j),$$

(3)

where $c_{ij} : S \times S \rightarrow \mathbb{R}$ denotes the payoff of $P_i$ playing with its neighbor $P_j, j \in U(i)$.

The strategy updating rule is: For each $i = 1, 2$, $P_i(t+1)$ is updated by the best strategy from strategies of its neighbors in $U(i)$ at time $t$. Precisely, if $j^* = \arg \max_{j \in U(i)} c_j(P_j, P_k | k \in U(j))$, then $P_i(t+1) = P_{j^*}(t)$. When the neighbors with maximum payoff are not unique, say, $\arg \max_{j \in U(i)} c_j(P_j, P_k | k \in U(j)) := \{j_1^*, \cdots, j_r^*\}$, we choose $j^* = \min\{j_1^*, \cdots, j_r^*\}$.

According to the strategy updating rule, we obtain the following 3-valued logical network:

$$P_i(t+1) = f_i(P_1(t), P_2(t), P_3(t), P_4(t)),$$

(4)

where $f_i, i = 1, 2, 3, 4$ are 3-valued logical functions, which can be uniquely determined by the strategy updating rule.
Introduction

Semi-Tensor Product of Matrices

Prof. Daizhan Cheng developed the semi-tensor product (STP) of matrices for the analysis and control of logical networks.*

The STP of $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is

$$ A \bowtie B = (A \otimes I_{\alpha \over n})(B \otimes I_{\alpha \over p}), \quad (5) $$

where $\alpha = lcm(n, p)$ denotes the least multiple of $n$ and $p$.

The main advantage of STP: converting a logical network into a (bi)linear form, which makes a bridge between logical networks and classic control theory.

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Semi-Tensor Product of Matrices

Given a logical function \( f : \mathcal{D}^s \mapsto \mathcal{D} \). There exists a unique structural matrix \( M_f \in \mathcal{L}_{2 \times 2^s} \) such that

\[
f(x_1, x_2, \cdots, x_s) = M_f \bigotimes_{i=1}^s x_i, \quad x_i \in \Delta := \{\delta_2^1, \delta_2^2\}. \tag{6}
\]

An algebraic state space representation approach is established for logical networks.

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Semi-Tensor Product of Matrices

Algebraic form of (1) is

\[ x(t + 1) = Lx(t), \]

where

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Semi-Tensor Product of Matrices

- The main advantage of STP: By the STP method, a logical expression can be converted into a linear (bilinear) form.

- Recent Results:

  - Analysis and Control of BNs: Controllability, Observability, Realization, Identification, Disturbance Decoupling, Optimal Control, etc.;
  - Model Generalization*: Delayed Logical Networks, Probabilistic Logical Networks, Asynchronous Logical Networks, Switched Logical Networks, Logical Networks with Impulsive Effect.
  - Applications*: Fault Detection of Circuits, Graph Coloring, Game Theory, Fuzzy Control, Finite Automata, Nonlinear Feedback Shift Register, Smart Grid, Vehicle Control, etc.

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Controllability and Observability of BNs

- **Controllability** means the reachability from any initial state to any terminal state, which provides a reachable set approach to the control design of BNs.

- **Observability** means to distinguish any two different initial states from a piece of output trajectories, which is important for identification and observer design.

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Controllability and Observability of BNs

Controllability and observability of BNs are two fundamental properties. Controllability means the reachability from any initial state to any terminal state, which provides a reachable set approach to the control design of BNs.

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Recent Works on Controllability

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Recent Works on Observability

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Set Controllability of BNs

Set controllability depicts the reachability from the family of initial sets to the family of destination sets.

Applications of set controllability: observability, set stabilization, partial stabilization, output tracking, synchronization, consensus, etc.

The key of applying set controllability is to properly construct the family of initial sets and the family of destination sets.


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Recent Works on Output Tracking

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Notations

1. \( \mathcal{D} := \{1, 0\} \).

2. \( \Delta_n := \{\delta^k_n : k = 1, \ldots, n\} \), where \( \delta^k_n \) denotes the \( k \)-th column of \( I_n \). \( \Delta := \Delta_2 \).

3. An \( n \times t \) logical matrix \( M = [\delta^{i_1}_n \delta^{i_2}_n \cdots \delta^{i_t}_n] \) is briefly denoted by \( M = \delta_n[i_1 \ i_2 \ \cdots \ i_t] \). The set of \( n \times t \) logical matrices is denoted by \( \mathcal{L}_{n \times t} \).

4. Given \( A \in \mathbb{R}^{m \times n} \), \( \text{Col}_i(A) \), \( \text{Row}_i(A) \) and \( (A)_{i,j} \) denote the \( i \)-th column, the \( i \)-th row and the \( (i,j) \)-th element of \( A \), respectively. \( \text{Col}_i(A) = A\delta^i_n \), \( \text{Row}_i(A) = (\delta^i_m)^\top A \).

5. Given \( A \in \mathbb{R}^{n \times mp} \), denote the \( i \)-th \( n \times p \) block of \( A \) by \( \text{Blk}_i(A) \).

6. \( \neg, \lor \) and \( \land \) denote “Negation”, “Disjunction” and “Conjunction”, respectively.
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Set Controllability of BNs

Boolean Control Networks

Consider the following Boolean control network (BCN):

\[
\begin{align*}
X(t+1) &= f(U(t), X(t)), \\
Y(t) &= h(X(t)).
\end{align*}
\]  

(7)

\(X(t) = (x_1(t), \cdots, x_n(t)) \in \mathcal{D}^n, \ U(t) = (u_1(t), \cdots, u_m(t)) \in \mathcal{D}^m, \ \text{and} \ Y(t) = (y_1(t), \cdots, y_p(t)) \in \mathcal{D}^p\) denote the states, control inputs, and outputs of system (7) at time \(t\), respectively.

\(f : \mathcal{D}^{m+n} \rightarrow \mathcal{D}^n\) and \(h : \mathcal{D}^n \rightarrow \mathcal{D}^p\) are logical mappings.
Set Controllability of BNs

Definition of Controllability

**Definition 1 (D. Cheng et al., Springer, 2011)**

Consider BCN (7) and given $X_0, X_d \in \mathcal{D}^n$.

1. $X_d$ is said to be reachable from $X_0$ at the $T$-th step, $T \in \mathbb{Z}_+$, if one can find a control sequence $\{U(t) \in \mathcal{D}^m : t = 0, \cdots, T - 1\}$ such that $X(T; X_0, U) = X_d$;

2. $X_d$ is said to be reachable from $X_0$, if there exists $T \in \mathbb{Z}_+$ such that $X_d$ is reachable from $X_0$ at the $T$-th step;

3. BCN (7) is said to be controllable at $X_0$, if for any given $X_d \in \mathcal{D}^n$, $X_d$ is reachable from $X_0$;

4. BCN (7) is said to be controllable, if it is controllable at any $X_0 \in \mathcal{D}^n$. 
Set Controllability of BNs

Algebraic Form of BCNs

Using the vector form of Boolean variables, that is, \(2 - i \sim \delta_i^2, i = 1, 2\), BCN (7) can be converted into the following algebraic form:

\[
\begin{aligned}
    x(t + 1) &= Lu(t)x(t), \\
    y(t) &= Hx(t),
\end{aligned}
\]

(8)

where \(x(t) = \bigotimes_{i=1}^n x_i(t), u(t) = \bigotimes_{i=1}^m u_i(t), y(t) = \bigotimes_{i=1}^p y_i(t),\)

\(L \in \mathcal{L}_{2^n \times 2^n + m}\) is the state transition matrix, and \(H \in \mathcal{L}_{2p \times 2^n}\) is the output matrix.
Set Controllability of BNs

Controllability Matrix

◊ Controllability matrix:

\[
C := \sum_{i=1}^{2^n} M^i.
\]

where \( M := \sum_{i=1}^{2^m} L\delta_{2m}^i \) is the one-step state transition matrix.

Remark 1

When defining the controllability matrix, the matrix addition and matrix product can be replaced by Boolean addition \( (+_B) \) and Boolean product \( (\times_B) \), respectively\(^a\).

Set Controllability of BNs

Controllability Criterion

**Theorem 1** (Y. Zhao et al., Syst. Control Lett., 2010)

Consider BCN (8).

(i). \( x_d = \delta^\varphi_{2^n} \) is reachable from \( x_0 = \delta^\theta_{2^n} \) at the \( T \)-th step, \( T \in \mathbb{Z}_+ \), if and only if \((M^T)_{\varphi, \theta} > 0\).

(ii). \( x_d = \delta^\varphi_{2^n} \) is reachable from \( x_0 = \delta^\theta_{2^n} \), if and only if \((C)_{\varphi, \theta} > 0\).

(iii). System (8) is controllable at \( x_0 = \delta^\theta_{2^n} \), if and only if \( Col_\theta(C) > 0\).

(iv). System (8) is controllable, if and only if \( C > 0 \).
Set Controllability of BNs

Proof of (i). in Theorem 1

✓ When $T = 1$, by Definition 1, $x_d = \delta_{2^n}^\varphi$ is reachable from $x_0 = \delta_{2^n}^\theta$ at the first step, if and only if there exists $u(0) := \delta_{2^m}^\mu$ such that $\delta_{2^n}^\varphi = L\delta_{2^m}^\mu \delta_{2^n}^\theta$. Then, we have

$$(M)_{\varphi,\theta} = \sum_{i=1}^{2^m} (L\delta_{2^m}^i)_{\varphi,\theta} \geq (L\delta_{2^m}^\mu)_{\varphi,\theta} = 1 > 0.$$  

✓ Assume that the conclusion holds for $T = s \geq 1$. When $T = s + 1$, since a path from $x_0$ to $x_d$ at the $(s + 1)$-th step can always be considered as a path from $x_0$ to $\bar{x}_0 = \delta_{2^n}^\gamma$ at the $s$-th step and then from $\bar{x}_0$ to $x_d$ at the first step, we have $(M^s)_{\gamma,\theta} > 0$ and $(M)_{\varphi,\gamma} > 0$. Therefore,

$$(M^{s+1})_{\varphi,\theta} = \sum_{k=1}^{2^n} (M)_{\varphi,k} (M^s)_{k,\theta} \geq (M)_{\varphi,\gamma} (M^s)_{\gamma,\theta} > 0.$$  

✓ By induction, the conclusion is true for any $T \in \mathbb{Z}_+$. 
Set Controllability of BNs

Definition of Set Controllability

Given the family of initial sets $P_0$ and the family of destination sets $P_d$ respectively as follows:

$$P_0 := \{ s^0_1, s^0_2, \ldots, s^0_\alpha \} \subseteq 2^{\Delta_{2n}}, \quad s^0_j \neq \emptyset, j = 1, 2, \ldots, \alpha;$$

$$P_d := \{ s^d_1, s^d_2, \ldots, s^d_\beta \} \subseteq 2^{\Delta_{2n}}, \quad s^d_i \neq \emptyset, i = 1, 2, \ldots, \beta.$$


BCN (8) with $P_0$ and $P_d$ is

1. set controllable from $s^0_j \in P_0$ to $s^d_i \in P_d$, if there exist $x_0 \in s^0_j$ and $x_d \in s^d_i$, such that $x_d$ is reachable from $x_0$;
2. set controllable at $s^0_j$, if for any $s^d_i \in P_d$, it is set controllable from $s^0_j$ to $s^d_i$;
3. set controllable, if it is set controllable at any $s^0_j \in P_0$. 
Set Controllability of BNs

Set Controllability Matrix

Set controllability matrix:

\[ C_S := J_d^T C J_0. \]

\( J_0 \) (\( J_d \)) is the initial (destination) index matrix defined as

\[ J_0 := [V(s_1^0) \ V(s_2^0) \ \cdots \ V(s_\alpha^0)], \]
\[ J_d := [V(s_1^d) \ V(s_2^d) \ \cdots \ V(s_\beta^d)]. \]

\( V(s) \in \mathbb{R}^{2n} \) with \( (V(s))_i := \begin{cases} 1, & \delta_{2n}^i \in s, \\ 0, & \delta_{2n}^i \notin s \end{cases} \) is the index vector of \( s \subseteq \Delta_{2n} \).
Set Controllability of BNs

Set Controllability Criterion


BCN (8) with $P_0$ and $P_d$ is

(i). set controllable from $s_j^0$ to $s_i^d$, if and only if $(C_S)_{i,j} > 0$.

(ii). set controllable at $s_j^0$, if and only if $Col_j(C_S) > 0$.

(iii). set controllable, if and only if $C_S > 0$.

**Remark 2**

Controllability is a spacial kind of set controllability. If

$$P_0 = P_d = \left\{ \{ \delta_1 \}, \cdots, \{ \delta_{2^n} \} \right\},$$

the set controllability degrades to the controllability defined in Definition 1.
Set Controllability of BNs

An Example

Given $P_0 = \left\{ \{\delta_8^2, \delta_8^3\}, \{\delta_8^1, \delta_8^5, \delta_8^7\} \right\}$ and $P_d = \left\{ \{\delta_8^4\}, \{\delta_8^2, \delta_8^8\} \right\}$. Consider the set controllability of BCN (8) with $L = \delta_8[3 \ 5 \ 1 \ 3 \ 1 \ 5 \ 6 \ 3 \ 7 \ 4 \ 1 \ 3 \ 1 \ 5 \ 6 \ 3]$.

Controllability matrix:

$$C := \sum_{t=1}^{8} \left( \sum_{i=1}^{2} L \delta_2^i \right)^t = \begin{bmatrix}
230 & 170 & 110 & 220 & 110 & 220 & 120 & 220 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
55 & 92 & 110 & 62 & 110 & 60 & 120 & 62 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 73 & 120 & 48 & 120 & 50 & 100 & 48 \\
110 & 84 & 60 & 120 & 60 & 120 & 50 & 120 \\
55 & 90 & 110 & 60 & 110 & 60 & 120 & 60 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

Initial and destination index matrices:

$$J_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top, \quad J_d = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$  

Set controllability matrix:

$$C_S = J_d^\top C J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Thus, BCN (8) is not set controllable with respect to $P_0$ and $P_d$. 
Observability of BNs

Definition of Observability

**Definition 3 (Y. Zhao et al., Syst. Control Lett., 2010)**

BCN (8) is **observable**, if for any two distinct states $x_0, \bar{x}_0$, there exists a control sequence $\{u_0, u_1, \cdots, u_{p-1}\}, p \in \mathbb{Z}_+$, such that the corresponding output sequences $(y_0, y_1, \cdots, y_p) \neq (\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_p)$.

**Remark 3**

K. Zhang and L. Zhang 2016\(^a\) discussed **four different definitions of observability** and pointed out that **Definition 3 is the most sharp one**. Thus, we **take Definition 3 as the standard one** and concentrate on this definition. An interesting topic is to explore other definitions via set controllability approach.

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Observability of BNs

Construct Dual System

- Split the product state space $\Delta_{2^n} \times \Delta_{2^n}$ into a partition of three components as

$$D = \{zx : z = x\}, \quad \Theta = \{zx : z \neq x, Hz = Hx\}, \quad \Xi = \{zx : z \neq x, Hz \neq Hx\}.$$  

- Using algebraic form (8), construct a dual system as

$$\begin{align*}
  z(t+1) &= Lu(t)z(t), \\
x(t+1) &= Lu(t)x(t).
\end{align*}$$  \hspace{3cm} (9)

Remark 4

By classifying the pairs of states, Cheng et al. 2016\textsuperscript{a} proposed an effective criterion for observability via constructing observability matrix. However, in order to obtain the observability matrix, one needs to proceed an iterative algorithm.

Observability of BNs

Observability Criterion

Construct the family of initial sets $P_0$ and the family of destination sets $P_d$ respectively as follows:

\[ P_0 := \bigcup_{zx \in \Theta} \{\{zx\}\}; \quad (10) \]

\[ P_d := \{\Xi\}. \quad (11) \]

Theorem 3 (D. Cheng et al., Syst. Control Lett., 2018)

BCN (8) is observable, if and only if system (9) is set controllable with respect to $P_0$ and $P_d$, which are defined in (10) and (11), respectively.
An Example

Example 1

Consider the reduced model for the lac operon in the bacterium *Escherichia coli*\(^a\):

\[
\begin{align*}
x_1(t + 1) &= \neg u_1(t) \land (x_2(t) \lor x_3(t)), \\
x_2(t + 1) &= \neg u_1(t) \land u_2(t) \land x_1(t), \\
x_3(t + 1) &= \neg u_1(t) \land (u_2(t) \lor (u_3(t) \land x_1(t))), \\
\end{align*}
\]

(12)

where \(x_1, x_2\) and \(x_3\) represent lac mRNA, lactose in high and medium concentrations, respectively; \(u_1, u_2\) and \(u_3\) are extracellular glucose, high and medium extracellular lactose, respectively.

Observability of BNs

An Example

Assume that the outputs are

\[
\begin{align*}
y_1(t) &= x_1(t) \land \neg x_2(t) \land x_3(t), \\
y_2(t) &= (x_1(t) \land \neg x_3(t)) \lor \{\neg x_1(t) \\
&\quad \lor [x_2(t) \lor (\neg x_2(t) \land \neg x_3(t))]\}, \\
y_3(t) &= (x_1(t) \land \neg x_2(t) \land x_3(t)) \lor (\neg x_1(t) \land x_3(t)).
\end{align*}
\]

(13)

The algebraic form of BCN (12) with outputs (13) is

\[
\begin{align*}
x(t+1) &= Lu(t)x(t), \\
y(t) &= Hx(t),
\end{align*}
\]

where

\[
L = \delta_8[8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8]
\]

and

\[
H = \delta_8[8\ 6\ 3\ 6\ 5\ 6\ 7\ 6].
\]
Observability of BNs

An Example

It is easy to figure out that

\[ \Theta = \{ \delta_{64}^{12}, \delta_{64}^{14}, \delta_{64}^{16}, \delta_{64}^{26}, \delta_{64}^{30}, \delta_{64}^{32}, \delta_{64}^{42}, \delta_{64}^{44}, \delta_{64}^{48}, \delta_{64}^{58}, \delta_{64}^{60}, \delta_{64}^{62} \}, \]

\[ \Xi = \Delta_{64} \setminus \left( \Theta \cup \{ \delta_{64}^{1}, \delta_{64}^{10}, \delta_{64}^{19}, \delta_{64}^{28}, \delta_{64}^{37}, \delta_{64}^{46}, \delta_{64}^{55}, \delta_{64}^{64} \} \right). \]

Construct a dual system as

\[ \begin{cases} 
  z(t + 1) = Lu(t)z(t), \\
  x(t + 1) = Lu(t)x(t). 
\end{cases} \quad (14) \]

Setting \( w(t) = z(t)x(t) \), we have

\[ w(t + 1) = \bar{L}u(t)w(t), \]

where \( \bar{L} = \delta_{64}[64 \ 64 \ 64 \ \cdots \ 60 \ 60 \ 60 \ 64] \).

The controllability matrix of system (14) can be calculated by

\[ \bar{C} := \sum_{j=1}^{64} \left( \sum_{i=1}^{8} \bar{L}\delta_{8}^{i} \right)^{j}. \]
An Example

Construct the family of initial sets $P_0$ and the family of destination sets $P_d$ respectively as follows:

$$P_0 := \bigcup_{zx \in \Theta} \{\{zx\}\},\ P_d := \{\Xi\}.$$

According to $P_0$ and $P_d$, we have

$$J_0 = \delta_{64}[12\ 14\ 16\ 26\ 30\ 32\ 42\ 44\ 48\ 58\ 60\ 62],\ J_d = \sum_{\delta_{64} \in \Xi} \delta^i_{64}.$$

Since

$$C_S = J_d^\top \bar{C}J_0 > 0,$$

system (14) is set controllable with respect to $P_0$ and $P_d$.

By virtue of Theorem 3, BCN (12) with outputs (13) is observable.
Outline

1. Introduction
2. Set Controllability of Boolean Networks
3. Observability of Boolean Networks
4. Output Tracking of Boolean Networks
5. Concluding Remarks
Output Tracking of BNs

Definition of Output Tracking

Definition 4 (H. Li et al., Automatica, 2015)
Given a constant reference signal $y^* \in \Delta_{2^p}$. BCN (8) is trackable with respect to $y^*$, if for any initial state $x_0 \in \Delta_{2^n}$, there exists $T \in \mathbb{Z}_+$ and a control sequence $\{u(t) : t \in \mathbb{N}\}$, such that $y(t; x_0, u) = y^*$ holds for any integer $t \geq T$.

For the given reference signal $y^* = \delta_{2^p}^\alpha$, define

$$\mathcal{O}(\alpha) := \{\delta_{2^n}^i \in \Delta_{2^n} : Col_i(H) = y^*\}, \quad (15)$$

and presuppose $\mathcal{O}(\alpha) \neq \emptyset$. 
Output Tracking of BNs

Output Tracking Criterion-Reachable Set Approach

- For $S \subseteq \Delta_{2^n}$, $S \neq \emptyset$ and $k \in \mathbb{Z}_+$, define

$$R_k(S) = \left\{ x_0 \in \Delta_{2^n} : \text{there exists } \{u(t) \in \Delta_{2^m} : t = 0, \ldots, k-1 \} \text{ such that } x(k; x_0, u) \in S \right\}.$$ 

Theorem 4 (H. Li et al., Automatica, 2015)

The output of BCN (8) tracks the reference signal $y^* = \delta_{2^p}^\alpha$ by a state feedback control, if and only if there exist a nonempty set $S \subseteq \mathcal{O}(\alpha)$ and an integer $1 \leq \tau \leq 2^n$, such that

$$\begin{cases} S \subseteq R_1(S), \\ R_{\tau}(S) = \Delta_{2^n}. \end{cases}$$

(16)
Output Tracking of BNs

Output Tracking Criterion-Reachable Set Approach

- Consider BCN (8) with \( L = \delta_{2^m} [i_1 i_2 \cdots i_{2m+n}] \). Suppose that there exist a nonempty set \( S \subseteq \mathcal{O}(\alpha) \) and an integer \( 1 \leq \tau \leq 2^n \) such that (16) holds.

- For each integer \( 1 \leq j \leq 2^n \), there exists a unique integer \( 1 \leq k_j \leq \tau \) such that \( \delta_j \in \mathcal{R}_{k_j}^\circ(S) \), where \( \mathcal{R}_{k_j}^\circ(S) = \mathcal{R}_k(S) \setminus \mathcal{R}_{k-1}(S) \), \( \mathcal{R}_0(S) := \emptyset \).

- Let \( 1 \leq p_j \leq 2^m \) be such that \( 1 \leq l \leq 2^{m+n} \) and

\[
\begin{align*}
\delta_{2^n}^l &\in S, & k_j &= 1, \\
\delta_{2^n}^l &\in \mathcal{R}_{k_j-1}(S), & 2 \leq k_j &\leq \tau,
\end{align*}
\]

where \( l = (p_j - 1)2^n + j \).

Corollary 1 (H. Li et al., Automatica, 2015)

The state feedback based output tracking control can be designed as \( u(t) = Kx(t) \) with \( K = \delta_{2^m} [p_1 p_2 \cdots p_{2^n}] \).
Output Tracking of BNs

Output Tracking Criterion-Set Controllability Approach

**Definition 5**

Consider BCN (8) and given a nonempty set \( S \subseteq \Delta_{2n} \).

1. \( S \) is said to be a **control invariant subset** of BCN (8), if for any \( x_0 \in S \), there exists a control \( u_{x_0} \in \Delta_{2m} \), such that \( x(1; x_0, u_{x_0}) \in S \).

2. \( S \) is said to be **globally reachable**, if for any \( x_0 \in \Delta_{2n} \), there exist \( T \in \mathbb{Z}_+ \) and a control sequence \( \{u(t) : t = 0, 1, \cdots, T - 1\} \), such that \( x(T; x_0, u) \in S \).
Output Tracking of BNs

Output Tracking Criterion-Set Controllability Approach

⋄ Construct $P_0 = \left\{ \{\delta_1^1\}, \cdots, \{\delta_{2^n}^{2^n}\} \right\}$.

⋄ For $S \subseteq \Delta_{2^n}$ and $S \neq \emptyset$, construct

$$\bar{P}_0 = \{s^0_i : i = 1, \cdots, |S|\}, \; s^0_i = \{x^i \in S\}, \; P_d = \{S\}.$$ 

Theorem 5

The output of BCN (8) tracks the reference signal $y^* = \delta_{2^p}^\alpha$, if and only if there exists a nonempty set $S \subseteq O(\alpha)$ satisfying the following two conditions:

(i). $S$ is a control invariant subset of BCN (8) $\Leftrightarrow$ BCN (8) is one-step set controllable with respect to $\bar{P}_0$ and $P_d$ $\Leftrightarrow$ $J_d^\top M \bar{J}_0 > 0$, where $J_d$ and $\bar{J}_0$ are the index matrices of $P_d$ and $\bar{P}_0$, respectively;

(ii). $S$ is globally reachable $\Leftrightarrow$ BCN (8) is set controllable with respect to $P_0$ and $P_d$ $\Leftrightarrow$ $J_d^\top C J_0 > 0$, where $J_0$ is the index matrix of $P_0$. 
Recall Example 1. Assume that the outputs are

\[ \begin{align*}
  y_1(t) &= x_1(t), \\
  y_2(t) &= x_2(t).
\end{align*} \]

Our objective is to verify whether or not BCN (12) is trackable with respect to \( Y^* = (1, 0) \) via set controllability approach.

- Using the vector form of Boolean variables, we have

\[ H = \delta_4[1 \ 1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4], \quad y^* = \delta_4^2. \]

- According to (15), one can calculate that

\[ \mathcal{O}(2) = \{\delta_8^3, \delta_8^4\}. \]
Output Tracking of BNs

An Example

◊ Construct

\[ P_0 = \left\{ \{\delta_8^1\}, \cdots, \{\delta_8^8\} \right\}, \quad \bar{P}_0 = P_d = \{\{\delta_8^3\}\} \].

Correspondingly, we have \( J_0 = \Delta_8, \bar{J}_0 = J_d = \delta_8^3 \).

◊ On one hand, since

\[ J_d^\top M \bar{J}_0 = 1 > 0, \]

BCN (12) is one-step set controllable with respect to \( \bar{P}_0 \) and \( P_d \).

◊ On the other hand, since

\[ J_d^\top C J_0 > 0, \]

BCN (12) is set controllable with respect to \( P_0 \) and \( P_d \).

◊ By virtue of Theorem 5, \( S = \{\delta_8^3\} \subseteq \mathcal{O}(2) \) is a control invariant subset of BCN (12). In addition, \( S \) is globally reachable. Therefore, the output of BCN (12) tracks the reference signal \( y^* = \delta_4^2 \), which coincides with the conclusion obtained in Li et al. 2015.
Concluding Remarks

1 Set controllability is a powerful tool to deal with observability and output tracking of BNs.

2 Further study: (i) Generalization of set controllability; (ii) Applications to networked evolutionary games*, discrete event systems* and finite-field networks*; (iii) Computational complexity reduction; (iv) Sampled-data control, event-triggered control, pinning control*.

*Y. Li, H. Li, X. Ding, Set stability of switched delayed logical networks with application to finite-field consensus, Automatica, 2020, 113: 108768.
Concluding Remarks

1. Set controllability is a powerful tool to deal with observability and output tracking of BNs.

2. Further study: (i) Generalization of set controllability; (ii) Applications to networked evolutionary games *, discrete event systems * and finite-field networks *; (iii) Computational complexity reduction; (iv) Sampled-data control, event-triggered control, pinning control *.

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* Y. Li, H. Li, X. Ding, Set stability of switched delayed logical networks with application to finite-field consensus, Automatica, 2020, 113: 108768.

Concluding Remarks

Function Perturbation Theory of Boolean Networks


✔ S. Wang, H. Li, Graph-based function perturbation analysis for observability of multi-valued logical networks, *IEEE Transactions on Neural Networks and Learning Systems*, minor revision.


Thanks!